Chapter 2

Lining Up Linear Equations

In This Chapter

- Establishing a game plan for solving linear equations
- Working through special rules for linear inequalities
- Making short work of absolute value equations and inequalities

The term *linear* has the word *line* buried in it, and the obvious connection is that you can graph many linear equations as lines. But linear expressions can come in many types of packages, not just equations or lines. In this chapter, you find out how to deal with linear equations, what to do with the answers in linear inequalities, and how to rewrite linear absolute value equations and inequalities so that you can solve them.

Getting the First Degree: Linear Equations

Linear equations feature variables that reach only the first degree, meaning that the highest power of any variable you solve for is 1. The general form of a linear equation with one variable is $ax + b = c$.

The one variable is the x. But, no matter how many variables you see, the common theme to linear equations is that each variable has only one solution or value that satisfies the equation when matched with constants or specific other variables.

Solving basic linear equations

To solve a linear equation, you isolate the variable on one side of the equation by adding the same number to both sides — or you can subtract, multiply, or divide the same number on both sides.

For example, you solve the equation $4x - 7 = 21$ by adding 7 to each side of the equation, to isolate the variable and the multiplier, and then dividing each side by 4, to leave the variable on its own:

$$4x - 7 + 7 = 21 + 7 \quad \rightarrow \quad 4x = 28$$
$$4x \div 4 = 28 \div 4 \quad \rightarrow \quad x = 7$$

When a linear equation has grouping symbols such as parentheses, brackets, or braces, you deal with any distributing across and simplifying within the grouping symbols before you isolate the variable. For example, to solve the equation $5x - [3(x + 2) - 4(5 - 2x) + 6] = 20$, you first distribute the 3 and –4 inside the brackets:

$$5x - [3x + 6 - 20 + 8x + 6] = 20$$

Then you combine the terms that combine and distribute the negative sign (–) in front of the bracket; it's like multiplying through by –1:

$$\begin{aligned} 5x - \left[11x - 8\right] &= 20 \\ 5x - 11x + 8 &= 20 \end{aligned}$$

Simplify again, and you can solve for x:

$$\begin{aligned} -6x + 8 &= 20 \\ -6x &= 12 \\ x &= -2 \end{aligned}$$

Eliminating fractions

The problem with fractions, like cats, is that they aren't particularly easy to deal with. They always insist on having their own way — in the form of common denominators before you can add or subtract. And division? Don't get me started!

The best way to deal with linear equations that involve variables tangled up with fractions is to get rid of the fractions. Your game plan is to multiply both sides of the equation by the least common denominator of all the fractions in the equation.

Solve $\frac{x+2}{5}+\frac{4x+2}{7}=\frac{9-x}{2}$ for x.

Multiply each term in the equation by 70 — the least common denominator (also known as the *least common multiple*) for fractions with the denominators 5, 7, and 2:

$$^{14}\cancel{70}\left(\frac{x+2}{\cancel{5}_1}\right)+{}^{10}\cancel{70}\left(\frac{4x+2}{\cancel{7}_1}\right)={}^{35}\cancel{70}\left(\frac{9-x}{\cancel{2}_1}\right)$$

Now you distribute the reduced numbers over each set of parentheses, combine the like terms, and solve for *x:*

$$\begin{aligned}14x+28+40x+20&=315-35x\\54x+48&=315-35x\\89x&=267\\x&=3\end{aligned}$$

Lining Up Linear Inequalities

Algebraic *inequalities* show comparative relationships between a number and an expression or between two expressions. In other words, you use inequalities for comparisons.

Inequalities in algebra are expressed by the comparisons *less than* (<), *greater than* (>), *less than or equal to* (≤), and *greater than or equal to* (≥). A linear equation containing one variable has only one solution, but a linear inequality can have an infinite number of solutions.

Here are the rules for operating on inequalities (you can replace the < symbol with any of the inequality symbols, and the rules will still hold):

- If $a < b$, then $a + c < b + c$ (adding any number).
- If $a < b$, then $a - c < b - c$ (subtracting any number).
- If $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$ (multiplying by any *positive* number).

- If $a < b$ and $c < 0$, then $a \cdot c > b \cdot c$ (multiplying by any *negative* number).
- If $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$ (dividing by any *positive* number).
- If $a < b$ and $c < 0$, then $\frac{a}{c} > \frac{b}{c}$ (dividing by any *negative* number).
- If $\frac{a}{c} < \frac{b}{d}$, then $\frac{c}{a} > \frac{d}{b}$ (reciprocating fractions).

You must not multiply or divide an inequality by 0.

Solving basic inequalities

To solve a basic linear inequality, first move all the variable terms to one side of the inequality and the numbers to the other. After you simplify the inequality down to a variable and a number, you can find out what values of the variable will make the inequality into a true statement.

EXAMPLE Solve $3x + 4 > 11 - 4x$ for x.

Add $4x$ and subtract 4 from each side: $7x > 7$.

Divide each side by 7: $x > 1$.

The sense stayed the same, because you didn't multiply or divide each side by a negative number.

The rules for solving linear equations also work with inequalities — somewhat. Everything goes smoothly until you try to multiply or divide each side of an inequality by a negative number.

REMEMBER When you multiply or divide each side of an inequality by a negative number, you have to *reverse the sense* (change < to >, or vice versa) to keep the inequality true.

Solve the inequality $4(x - 3) - 2 \geq 3(2x + 1) + 7$ for x.

Distributing, you get: $4x - 12 - 2 \geq 6x + 3 + 7$.

Simplifying: $4x - 14 \geq 6x + 10$.

Now subtract $6x$ and add 14: $-2x \geq 24$.

Divide each side by –2, reversing the sense: $x \leq -12$.

Introducing interval notation

Much of higher mathematics uses interval notation instead of inequality notation. Interval notation is thought to be quicker and neater than inequality notation. Interval notation uses parentheses, brackets, commas, and the infinity symbol to bring clarity to the murky inequality waters.

To use *interval notation* when describing a set of numbers:

- You order any numbers used in the notation with the smaller number to the left of the larger number.
- You indicate "or equal to" by using a bracket.
- If the solution doesn't include the end number, you use a parenthesis.
- When the interval doesn't end (it goes up to positive infinity or down to negative infinity), use $+\infty$ or $-\infty$, whichever is appropriate, and a parenthesis.

Here are some examples of inequality notation and the corresponding interval notation:

Inequality Notation	***Linear Notation***
$x < 3$	$(-\infty, 3)$
$x \geq -2$	$[-2, \infty)$
$4 \leq x < 9$	$[4, 9)$
$-3 < x < 7$	$(-3, 7)$

Solve the inequality $-8 \le 3x - 5 < 10$.

Add 5 to each of the three sections and then divide each section by 3:

$$\begin{array}{rcccl} -8 & \le & 3x - 5 & < & 10 \\ \underline{+5} & & \underline{+5} & & \underline{+5} \\ -3 & \le & 3x & < & 15 \\ \frac{-3}{3} & \le & \frac{3x}{3} & < & \frac{15}{3} \\ -1 & \le & x & < & 5 \end{array}$$

You write the answer, $-1 \le x < 5$, in interval notation as $[-1, 5)$.

Absolute Value: Keeping Everything in Line

When you perform an *absolute value operation,* you're not performing surgery at bargain-basement prices; you're taking a number inserted between the absolute value bars, $|a|$, and recording the distance of that number from 0 on the number line. For example, $|3| = 3$, because 3 is three units away from 0. On the other hand, $|-4| = 4$, because -4 is four units away from 0.

The absolute value of a is defined as

$$|a| = \begin{cases} a \text{ if } a \ge 0 \\ -a \text{ if } a < 0 \end{cases}$$

You read the definition as follows: "The absolute value of a is equal to a, itself, if a is positive or 0; the absolute value of a is equal to the *opposite* of a if a is negative."

Solving absolute value equations

A linear absolute value equation is an equation that takes the form $|ax + b| = c$. To solve an absolute value equation in this linear form, you have to consider both possibilities: $ax + b$ may be positive, or it may be negative.

To solve for the variable x in $|ax + b| = c$, you solve both $ax + b = c$ and $ax + b = -c$.

Solve the absolute value equation $3|4 - 3x| + 7 = 25$.

First, you have to subtract 7 from each side of the equation and then divide each side by 3:

$$\begin{aligned} 3|4-3x|+7&=25\\ 3|4-3x|&=18\\ |4-3x|&=6 \end{aligned}$$

Then you can apply the rule for changing the absolute value equation to two linear equations:

$$\begin{aligned} 4-3x&=6 & \qquad 4-3x&=-6\\ -3x&=2 & -3x&=-10\\ x&=-\frac{2}{3} & x&=\frac{10}{3} \end{aligned}$$

Seeing through absolute value inequality

An absolute value inequality contains an absolute value, $|a|$, and an inequality: $<$, $>$, $\le$, or $\ge$.

To solve an absolute value inequality, you have to change the form from absolute value to just plain inequality.

- To solve for x in $|ax + b| < c$, you solve $-c < ax + b < c$.
- To solve for x in $|ax + b| > c$, you solve $ax + b > c$ and $ax + b < -c$.

Solve the absolute value inequality: $|2x - 1| \le 5$.

Rewrite the inequality: $-5 \le 2x - 1 \le 5$.

Next, add 1 to each section: $-4 \le 2x \le 6$.

Divide each section by 2: $-2 \le x \le 3$.

You can write the solution in interval notation as $[-2, 3]$.

Solve $|7 - 2x| > 11$ for x.

Rewrite the absolute value inequality as two separate inequalities: $7 - 2x > 11$ and $7 - 2x < -11$.

When solving the two inequalities, be sure to remember to switch the sign when you divide by –2:

$$\begin{aligned} 7 - 2x &> 11 \\ -2x &> 4 \\ x &< -2 \end{aligned} \qquad\qquad \begin{aligned} 7 - 2x &< -11 \\ -2x &< -18 \\ x &> 9 \end{aligned}$$

The solution $x < -2$ or $x > 9$, in interval notation, is $(-\infty, -2)$ or $(9, \infty)$.

Don't write the solution $x < -2$ or $x > 9$ as $9 < x < -2$. If you do, you indicate that some numbers can be bigger than 9 *and* smaller than –2 at the same time, which just isn't so.